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2007 J. Phys. A: Math. Theor. 40 9331

(http://iopscience.iop.org/1751-8121/40/31/011)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 9331–9342 doi:10.1088/1751-8113/40/31/011

The Tresse theorem and differential invariants for the nonlinear Schrödinger equation

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Received 19 March 2007, in final form 14 June 2007
Published 19 July 2007
Online at stacks.iop.org/JPhysA/40/9331

Abstract
In the present paper by using the Tresse theorem we describe a method of construction of all invariants and the differential invariants for a given Lie group, which means invariants containing derivatives of any order. Some important examples from analysis, geometry and physics are presented. In particular, invariants for the nonlinear Schrödinger equation will be investigated.

PACS number: 11.30.−j
Mathematics Subject Classification: 53A55, 34C14

1. Introduction

In the group theoretical analysis of differential equations and physical applications, expressions that are invariant with respect to some group of transformations, such as rotation, Lorentz or Poincaré groups, as well as their subgroups and extensions, play a prominent role in the study and establishment of conservations laws. Such significant functions for example are distance, pseudodistance, curvature, torsion, etc [1]. The essential feature of some of them, describing the physical phenomena in the space $(x_1, x_2, \ldots, x_n, u) \in \mathbb{R}^{n+1}$, where $u = u(x_1, x_2, \ldots, x_n)$, is the fact that they contain derivatives of the function u. In many cases there are the first- and second-order derivatives, but generally one can construct invariants containing derivatives of any order.

In this paper, we use the Tresse theorem [2–4] in order to describe a method of constructing a basis of invariants and the so-called invariant differentiation operators for a given Lie group G. Then, by using the basis and invariant differentiation operators one can obtain any invariant of G, i.e. any invariant containing derivatives of any order. We apply the Tresse theorem to the most important groups of transformations, that is a rotation group, the Lorentz groups in several cases according to the dimension and to the symmetry group of the nonlinear Schrödinger equation with arbitrary potential depending on $|\psi|$ and in particular to equation with cubic nonlinearity.

In this paper, we also analyse the differential invariants of nonlinear Schrödinger equations and establish which of them can be considered as fundamental, in the sense they are not obtained from other invariants by the application of the Tresse theorem.

The plan of this paper is as follows. In section 2, we give basic notations and definitions. In section 3, we formulate the Tresse theorem and some important properties of the differential invariants of a Lie group G. Several examples, illustrating an application of this theorem to the rotation and Lorentz groups, are presented. In section 4, having the equivalence algebra of nonlinear Schrödinger equation with variable potential, we construct differential invariants and invariant differentiation operators of symmetry algebra of this equation. Then, we investigate which invariants are fundamental, in the sense that they cannot be obtained from another invariants by using the Tresse theorem. In particular, we consider equation with cubic nonlinearity. Conclusions are presented in section 5.

Differential invariants and their applications for other important equations in physics, e.g. KdV, KP and Monge–Ampère equations, have been investigated by Olver, Chen, Pohjanpelto, Yehorchenko and Nutku, Sheftel in [5–8]. The differential invariants and the so-called semi-invariants of the generalized Schrödinger equation by using equivalence transformations have been described by Senthilvelan, Torrisi and Valenti in [9].

2. Basic notations and definitions

We use the following notation: u denotes a real function $u(x_1, \ldots, x_n)$, u_{x_i} denotes the partial derivative $\frac{\partial u}{\partial x_i}$, $\underset{k}{u}$ denotes the set of all partial derivatives of the order k of a function u and D_i denotes the operator of the full differentiation over x_i.

Let consider the action of a Lie group G in the space of variables $(x_1, x_2, \ldots, x_n, u)$, where u is a dependent variable and corresponding infinitesimal generator $X = \xi^i \partial_{x_i} + \eta \partial_u$ in the Lie algebra of a Lie group G.

We denote by $\underset{m}{X}$ the extension of the mth order of an operator X to the space $(x_1, x_2, \ldots, x_n, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$ and define it by the formula

$$\underset{m}{X} = X + \sum_{p=1}^{m} \zeta^{i_1,\ldots,i_p} \partial_{u_{x_{i_1,\ldots,i_p}}},$$

where coefficients $\zeta^{i_1,\ldots,i_p}$ are defined by

$$\zeta^{i_1,\ldots,i_p} = D_{i_1,\ldots,i_p}(\eta - u_{x_k}\xi^k) + u_{x_{i_1},\ldots,x_{i_p},x_{i_k}} \cdot \xi^k,$$

where the summation is over k, $(i_1, i_2, \ldots, i_m)$ being fixed, $\xi = \xi(x, u, \underset{1}{u}, \ldots, \underset{s}{u})$, $\eta = \eta(x, u, \underset{1}{u}, \ldots, \underset{s}{u})$.

Definition 2.1. *Let G be a Lie group of transformations with the parameter $a \in \mathbb{R}$, $f, g \in G$, $x \in \mathbb{R}^n$, $u = u(x_1, \ldots, x_n)$ and $\widetilde{x} = f(x, u, a)$, $\widetilde{u} = g(x, u, a)$:*

(a) A function $F(x, u)$ is called an invariant of G iff

$$\forall_{a\in\mathbb{R}} F(\widetilde{x}, \widetilde{u}) = F(x, u).$$

(b) An expression $F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$ is called a differential invariant (of the mth order) of the group G iff

$$\forall_{a\in\mathbb{R}} F(\widetilde{x}, \widetilde{u}, \underset{1}{\widetilde{u}}, \underset{2}{\widetilde{u}}, \ldots, \underset{m}{\widetilde{u}}) = F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}).$$

This is an invariant of the action of the group G extended to the space $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$.

(c) The general (or universal) differential invariant of the mth order is the set of all differential invariants from the order zero to the order m inclusive.

(d) *A maximal set of functionally independent invariants of the order $r \leqslant m$ of a Lie group G is called a functional basis of the mth-order differential invariants of G.*

(e) *Q is called an operator of the invariant differentiation, if for any differential invariant F of the group G the expression QF is also the differential invariant of the group G.*

For an infinitesimal generator X of the group G the infinitesimal invariance test has the form

$$\underset{m}{X}F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = 0.$$

We use these notions following Ovsiannikov [3] and Olver [10].

3. The Tresse theorem

In the theory of differential invariants for Lie groups the following theorem is fundamental.

Theorem 3.1 (Tresse, 1894) [2, 3]**.** *For a given Lie group G with r parameters, acting in the space (x, u), $x \in V \subset \mathbb{R}^n$, $u : V \to \mathbb{R}$, there exists a finite basis of functionally independent invariants and exist operators of the invariant differentiation Q_j such that arbitrary fixed order invariant of G can be obtained in a finite number of invariant differentiations and functional operations on invariants from the basis.*

This finite basis of invariants includes in the general differential invariant of the minimal order $s \geqslant 1$ such that

$$\forall_{(x,u)\in V\times\mathbb{R}} \qquad r = \operatorname{rank}[\xi(x, u), \eta(x, u), \zeta^1(x, u, \underset{1}{u}), \ldots, \zeta^{s-1}(x, u, \underset{1}{u}, \ldots, \underset{s-1}{u})]. \tag{3.1}$$

The number of operators of the invariant differentiation, independent of the field of invariants of G, is equal to n. They are defined by

$$Q_j = \lambda^i_j\,(x, u, \underset{1}{u}, \ldots, \underset{s}{u}) D_i, \tag{3.2}$$

where $\lambda_j = \left[\lambda^i_j\right]$ satisfies the commutation condition of operators Q_j and X_ν:

$$\underset{s}{X}_{\nu}\lambda_j = \lambda^i_j D_i(\xi_\nu), \tag{3.3}$$

where the summation is over $i = 1, \ldots, n$ and $X_\nu = \xi^i_\nu \partial_{x_i} + \eta_\nu \partial_u$, for $\nu = 1, \ldots, r$, are generators of the Lie algebra of the group G. In (3.1), $\xi(x, u) = [\xi_1, \ldots, \xi_r]^T$, $\xi_\nu = \left[\xi^1_\nu, \ldots, \xi^n_\nu\right]^T$, $\eta(x, u) = [\eta_1, \ldots, \eta_r]^T$.

Remark 3.1. If a group G acts in the space $(x_1, \ldots, x_n, u_1, \ldots, u_k) \in \mathbb{R}^{n+k}$, then the number of elements in a basis of the mth-order general invariant is given by the formula

$$R(m) = n + k \cdot \binom{n+m}{n} - r_m, \tag{3.4}$$

where r_m is a rank of the matrix of coefficients of the mth prolongation of operators X_ν.

Then the general mth-order invariant is expressed by arbitrary function depending on basic invariants $\Phi(\omega_1, \ldots, \omega_{R(m)})$.

Remark 3.2. If all invariants of the order s can be obtained from invariants of the order $s - 1$ by a finite number of invariant differentiation and functional operations then the basis of invariants from the Tresse theorem includes in the general invariant of the order $s - 1$. By analogy we have this property of reduction of the order for lower orders.

Fact 3.1 [3]

(a) *The set of invariant differentiation operators for a given Lie group G forms a Lie algebra over the field of invariants of G.*

(b) *If an operator Y commute with the infinite prolongation of any operator from Lie algebra of a Lie group G, then Y is an operator of the invariant differentiation for G.*
(c) *The basis of differential invariants from the Tresse theorem for a Lie group G uniquely defines G.*

We construct the following examples in order to illustrate the application of the Tresse theorem and the method of construction of invariants for important geometrical and physical groups of transformations. In all examples a denotes the group parameter and ω_{ki} denotes the differential invariants of the order k.

Example 3.1. The group of rotations in

$$\mathbb{R}^3 : \begin{cases} \widetilde{x} = x\cos a - y\sin a \\ \widetilde{y} = x\sin a + y\cos a, \quad r = 1, s = 1, \\ \widetilde{u} = u \end{cases}$$

with infinitesimal generator $X = -y\partial_x + x\partial_y$.

Invariants of the order zero satisfy the equation $X\omega = 0$ and they are $\omega_{01} = u, \omega_{02} = x^2 + y^2$.

Then $\underset{1}{X} = -y\partial_x + x\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y}$ and system (3.3) has the form

$$x\lambda_y - y\lambda_x - u_y\lambda_{u_x} + u_x\lambda_{u_y} = \lambda^1 \cdot [0, 1]^T + \lambda^2 \cdot [-1, 0]^T.$$

Hence, we find λ^1, λ^2 and the invariant differentiation operators

$$Q_1 = u_x D_x + u_y D_y, \qquad Q_2 = -u_y D_x + u_x D_y.$$

The basis of a general invariant of the first order, according to formula (3.4), consists of four elements. As there are two basic zeroth-order invariants, as remaining two invariants contain the first-order derivatives. Among all the first-order invariants we can choose two functionally independent ones in the following way:

$$\omega_{11} = u_x^2 + u_y^2 = Q_1(\omega_{01}), \qquad \omega_{12} = xu_x + yu_y = \tfrac{1}{2}Q_1(\omega_{02}),$$

and an auxiliary invariant $\omega_{13} = xu_y - yu_x$, which can be obtained in the following way:

$$\omega_{13} = \sqrt{\omega_{02} \cdot \omega_{11} - \omega_{12}^2}.$$

The second-order basic invariants are

$$\omega_{21} = Q_1(\omega_{13}) = xu_y u_{yy} - yu_x u_{xx} + (xu_x - yu_y)u_{xy},$$

$$\omega_{22} = \frac{1}{2}Q_1(\omega_{11}) = u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy},$$

$$\omega_{23} = \frac{Q_1(\omega_{12}) + Q_2(\omega_{13})}{\omega_{12}} = u_{xx} + u_{yy},$$

and an auxiliary invariant

$$\omega_{24} = Q_1(\omega_{12}) - \omega_{11} = xu_x u_{xx} + yu_y u_{yy} + (yu_x + xu_y)u_{xy}.$$

Then the Gaussian curvature has the invariant form

$$K = k_1 \cdot k_2 = \frac{u_{xx}u_{yy} - u_{xy}^2}{\left(1 + u_x^2 + u_y^2\right)^2} = \frac{\omega_{02} \cdot \omega_{22} \cdot \omega_{23} - \omega_{21}^2 - \omega_{24}^2}{\omega_{02} \cdot \omega_{11} \cdot (1 + \omega_{11})^2},$$

and the average curvature has the invariant form

$$H = \frac{z_{xx}\left(1 + z_y^2\right) - 2z_{xy}z_x z_y + z_{yy}\left(1 + z_x^2\right)}{2\left(1 + u_x^2 + u_y^2\right)^{3/2}} = \frac{\omega_{23} \cdot \omega_{11} - \omega_{22} + \omega_{23}}{2(1 + \omega_{11})^{3/2}}.$$

For this group we have the property that all differential invariants can be obtained from ω_{01}, ω_{02} by invariant differentiation and functional operations. It means that the most important information about invariancy is contained in invariants of the order zero, i.e. the radius and the invariant function u. Note that the coefficients of the first and second fundamental quadratic forms for a surface are not invariant.

Remark 3.3. For the group of rotations in $\mathbb{R}^3$ with the infinitesimal generators

$$X_1 = -y\partial_x + x\partial_y, \qquad X_2 = -u\partial_x + x\partial_u, \qquad X_3 = -u\partial_y + y\partial_u,$$

when the dependent variable u also rotates, the Gaussian curvature is not an invariant, because

$$\underset{2}{X}_2 K = 3u_x \cdot K, \qquad \underset{2}{X}_3 K = 3u_y \cdot K.$$

Remark 3.4. The Tresse theorem does not provide a real algorithm of obtaining all basic invariants of arbitrary order. It only states that there exists a basis of invariants and yields information about the quantity of elements in the basis. We find all invariants of the mth order using the condition $\underset{m}{X}F(x, u, \underset{1}{u}, \ldots, \underset{m}{u}) = 0$. Further, among these we choose $R(m)-R(m-1)$ functionally independent invariants.

Example 3.2. The Lorentz group in $(x, y, u) \in \mathbb{R}^3$ with the generator

$$X = y\partial_x + x\partial_y, \qquad r = 1, s = 1.$$

The base of invariants of the order zero has the form $u, x^2 - y^2$.
The first-order basic invariants are $u_x^2 - u_y^2, xu_x + yu_y$.
The invariant differentiation operators are

$$Q_1 = xD_x + yD_y, \qquad Q_2 = u_x D_x - u_y D_y,$$

and the first-order basic invariants can be obtained form the zeroth-order ones.

The invariant differentiation of variable u in the Q_1 direction gives the nth order invariant of the form

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \cdot u_{\underbrace{x\ldots x}_{k}\underbrace{y\ldots y}_{n-k}}.$$

Example 3.3. The Lorentz group in $(t, x, y, z, u) \in \mathbb{R}^5$ with the generators

$$X_1 = t\partial_x + x\partial_t, \qquad X_2 = t\partial_y + y\partial_t, \qquad X_3 = t\partial_z + z\partial_t,$$
$$X_4 = -y\partial_x + x\partial_y, \qquad X_5 = -z\partial_x + x\partial_z, \qquad X_6 = -z\partial_y + y\partial_z, \qquad r = 3, s = 3.$$

In this case, the rank of the first prolongation of generators is equal to 5, the rank of the second prolongation is equal to 6 and one needs to use the third prolongation.

Basic invariants of the order zero are $u, \quad t^2 - x^2 - y^2 - z^2$.

Basic first-order invariants are $u_t^2 - u_x^2 - u_y^2 - u_z^2, \quad tu_t + xu_x + yu_y + zu_z$.

The invariant differentiation operators Q_1, Q_2 have the form

$$Q_1 = tD_t + xD_x + yD_y + zD_z, \qquad Q_2 = u_t D_t - u_x D_x - u_y D_y - u_z D_z.$$

Further, using the notation $(t, x, y, z) = (x_0, x_1, x_2, x_3)$ one can write Q_3 and Q_4:

$$Q_3 = \sum_{i=0}^{3} \lambda^i D_{x_i}, \qquad Q_4 = \sum_{j=0}^{3} \lambda^j D_{x_j},$$

where for Q_3 we have

$$\lambda^0 = u_{x_0} \cdot u_{x_0x_0} - \sum_{k=1}^{3} u_{x_k} \cdot u_{x_0x_k}, \qquad \lambda^i = \sum_{k=1}^{3} u_{x_k} \cdot u_{x_ix_k} - u_{x_0} \cdot u_{x_0x_i}, \qquad i = 1, 2, 3,$$

and for Q_4 we have

$$\lambda^0 = \sum_{k=0}^{3} x_k \cdot u_{x_0x_k}, \qquad \lambda^j = -\sum_{k=0}^{3} x_k \cdot u_{x_jx_k}, \qquad j = 1, 2, 3.$$

Note that, as above, the first-order differential invariants can be obtained from zeroth-order ones by using Q_1, Q_2.

The differential invariants for the group $O(n)$ and its extensions, important in physical investigations, are described in [11].

4. Differential invariants for the nonlinear Schrödinger equation

4.1. The general case $\mathrm{i}\psi_t + \psi_{xx} + W(|\psi|) \cdot \psi = 0$

Let consider the nonlinear Schrödinger equation of the form

$$\mathrm{i}\psi_t + \psi_{xx} + W(|\psi|) \cdot \psi = 0, \tag{4.1}$$

where $x \in \mathbb{R}^1$ and W is an arbitrary smooth function.

The whole equivalence algebra and its properties for this equation are given in [12]. The group transformations are build for functions W, ψ, ψ^*, where ψ^* is the complex conjugation of ψ and $\psi\psi^* = |\psi|^2$. In this approach, we treat the potential W as a new variable which transforms itself, besides the transformations of ψ and ψ^*. If we assume that the variable W is invariant, then from the whole equivalence algebra (described in [12]) we can choose subalgebra of symmetry of this equation for any W. The generators of this subalgebra are in the form

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = \psi\partial_\psi - \psi^*\partial_{\psi^*}, \qquad X_4 = t\partial_x + \frac{\mathrm{i}}{2}x(\psi\partial_\psi - \psi^*\partial_{\psi^*}) \tag{4.2}$$

with commutation relations

$$\begin{aligned} &[X_1, X_2] = 0, \qquad &&[X_1, X_3] = 0, \qquad &&[X_1, X_4] = X_2, \\ &[X_2, X_3] = 0, \qquad &&[X_2, X_4] = \frac{\mathrm{i}}{2}X_3, \qquad &&[X_3, X_4] = 0. \end{aligned}$$

Hence, we observe that it is a solvable Lie algebra. We construct invariants of this algebra, noting that those invariants do not depend on variables t, x.

The invariant of the order zero is $\omega_0 = \psi\psi^* = |\psi|^2$.

The first-order invariants we find by integration of the characteristic equations system for X_4 and writing solutions in the invariant form with respect to X_3:

$$\omega_1 = \frac{\psi_x}{\psi} + \frac{\psi_x^*}{\psi^*}, \qquad \omega_2 = \frac{\psi_t}{\psi} - \mathrm{i} \cdot \left(\frac{\psi_x}{\psi}\right)^2, \qquad \omega_3 = \frac{\psi_t^*}{\psi^*} + \mathrm{i} \cdot \left(\frac{\psi_x^*}{\psi^*}\right)^2. \tag{4.3}$$

Note that ω_i, $i = 0, 1, 2, 3$, are functionally independent (over $\mathbb{R}$). Moreover, the matrix of coefficients of the first prolongation of operators from (4.2) has the rank equals 4. Hence, according formula (3.4) this is the maximal system of the first-order invariants of this algebra.

Remark 4.1. By using the differential invariants (4.3) one obtains the invariant, nonlinear equations of the first order, connected with the Schrödinger equation:

$$\mathrm{i}\psi_t + \frac{\psi_x^2}{\psi} = c_1\psi, \tag{4.4}$$

$$\mathrm{i}\psi_t^* - \frac{(\psi_x^*)^2}{\psi^*} = c_2\psi^*. \tag{4.5}$$

The physical properties of these equations, from the group-theoretical point of view, are also interesting.

The general second-order differential invariant of the Lie algebra (4.2) is easily seen to have ten generators. Among these, we have ω_i for $i = 0, 1, 2, 3$ and additional six containing derivatives of second order. We find the invariant differentiation operators Q_1, Q_2 by using the Tresse theorem and show that ω_i, $i = 0, 1, 2, 3$, and Q_1, Q_2 suffice to express all invariants containing derivatives of second order.

System (3.3) for operator X_1 has the form

$$\underset{2}{X_1}\cdot\begin{bmatrix}\lambda^1\\ \lambda^2\end{bmatrix} = \lambda^1\cdot D_t\begin{bmatrix}1\\ 0\end{bmatrix} + \lambda^2\cdot D_x\begin{bmatrix}1\\ 0\end{bmatrix},$$

that is

$$\begin{bmatrix}\lambda_t^1\\ \lambda_t^2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

We obtain that λ^1, λ^2 do not depend on t. By analogy from equation for X_2 they do not depend on x. Now let us write system (3.3) for X_3, X_4, setting that $\lambda^i(\psi, \psi^*, \psi_t, \psi_t^*, \psi_x, \psi_x^*)$:

$$\psi\cdot\begin{bmatrix}\lambda^1_{\psi}\\ \lambda^2_{\psi}\end{bmatrix} - \psi^*\cdot\begin{bmatrix}\lambda^1_{\psi^*}\\ \lambda^2_{\psi^*}\end{bmatrix} + \psi_t\cdot\begin{bmatrix}\lambda^1_{\psi_t}\\ \lambda^2_{\psi_t}\end{bmatrix} + \psi_x\cdot\begin{bmatrix}\lambda^1_{\psi_x}\\ \lambda^2_{\psi_x}\end{bmatrix} - \psi_t^*\cdot\begin{bmatrix}\lambda^1_{\psi_t^*}\\ \lambda^2_{\psi_t^*}\end{bmatrix} - \psi_x^*\cdot\begin{bmatrix}\lambda^1_{\psi_x^*}\\ \lambda^2_{\psi_x^*}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}, \tag{4.6}$$

$$\begin{aligned}&\frac{\mathrm{i}}{2}x\psi\cdot\begin{bmatrix}\lambda^1_{\psi}\\ \lambda^2_{\psi}\end{bmatrix} - \frac{\mathrm{i}}{2}x\psi^*\cdot\begin{bmatrix}\lambda^1_{\psi^*}\\ \lambda^2_{\psi^*}\end{bmatrix} + \left(\frac{\mathrm{i}}{2}x\psi_t - \psi_x\right)\cdot\begin{bmatrix}\lambda^1_{\psi_t}\\ \lambda^2_{\psi_t}\end{bmatrix} + \left(\frac{\mathrm{i}}{2}\psi + \frac{\mathrm{i}}{2}x\psi_x\right)\cdot\begin{bmatrix}\lambda^1_{\psi_x}\\ \lambda^2_{\psi_x}\end{bmatrix}\\ &\quad - \left(\frac{\mathrm{i}}{2}x\psi_t^* + \psi_x^*\right)\cdot\begin{bmatrix}\lambda^1_{\psi_t^*}\\ \lambda^2_{\psi_t^*}\end{bmatrix} - \left(\frac{\mathrm{i}}{2}\psi^* + \frac{\mathrm{i}}{2}x\psi_x^*\right)\cdot\begin{bmatrix}\lambda^1_{\psi_x^*}\\ \lambda^2_{\psi_x^*}\end{bmatrix} = \begin{bmatrix}0\\ \lambda^1\end{bmatrix}.\end{aligned} \tag{4.7}$$

Solving this system we obtain

$$\begin{bmatrix}\lambda^1\\ \lambda^2\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} \quad\vee\quad \begin{bmatrix}\lambda^1\\ \lambda^2\end{bmatrix} = \begin{bmatrix}\psi\psi^*\\ \mathrm{i}\end{bmatrix}(\psi\psi_x^* - \psi_x\psi^*).$$

Hence, the invariant differentiation operators are in the form

$$Q_1 = D_x, \qquad Q_2 = \psi\psi^* D_t + \mathrm{i}(\psi\psi_x^* - \psi_x\psi^*)D_x.$$

By fact (3.1) we have the invariant differentiation operator also in the form

$$Q_3 = [Q_1, Q_2] = (\psi_x\psi^* + \psi\psi_x^*)D_t + \mathrm{i}(\psi\psi_{xx}^* - \psi_{xx}\psi^*)D_x.$$

We observe that the coefficients of the invariant differentiation operators are real. Indeed $\psi\psi^* \in \mathbb{R}$ and we have

$$\overline{\mathrm{i}(\psi\psi_x^* - \psi_x\psi^*)} = -\mathrm{i}(\psi^*\psi_x - \psi_x^*\psi) = \mathrm{i}(\psi\psi_x^* - \psi_x\psi^*),$$
$$\overline{\psi_x\psi^* + \psi\psi_x^*} = \psi_x^*\psi + \psi^*\psi_x = \psi_x\psi^* + \psi\psi_x^*,$$
$$\overline{\mathrm{i}(\psi\psi_{xx}^* - \psi_{xx}\psi^*)} = -\mathrm{i}(\psi^*\psi_{xx} - \psi_{xx}^*\psi) = \mathrm{i}(\psi\psi_{xx}^* - \psi_{xx}\psi^*).$$

Hence, if $\omega = f(\psi, \psi^*, \ldots)$ is an invariant, then $\omega^* = \overline{f}(\psi, \psi^*, \ldots)$ is also an invariant. It is important that ω_1 can be gained using ω_0 by

$$\omega_1 = \frac{Q_1(\omega_0)}{\omega_0}.$$

Moreover,

$$\omega_2 + \omega_3 = \frac{Q_2(\omega_0)}{\omega_0^2}.$$

Nevertheless, one cannot obtain all differential invariants of the first order using ω_0 by invariant differentiation and functional operations. In this way, only two functionally independent expressions can be obtained while the basis of the first-order differential invariants contains three elements. Whereas additionally one separates variables ψ, ψ^* then it is possible to obtain three basic invariants of the first order.

We show that one obtains the basic second-order differential invariants from the first order ones by invariant differentiation, hence the basis from the Tresse theorem includes in the general invariant of the first order. In construction of functionally independent invariants, containing derivatives of the second order $\psi_{xx}, \psi_{xx}^*, \psi_{tx}, \psi_{tx}^*, \psi_{tt}, \psi_{tt}^*$, we use the property that Q_1, Q_2 commute with operators from algebra (4.2). Noting that

$$\underset{1\,4}{X}\left(\frac{\psi_x}{\psi}\right) = \frac{\mathrm{i}}{2}, \qquad \underset{1\,k}{X}\left(\frac{\psi_x}{\psi}\right) = 0, \qquad k = 1, 2, 3,$$

we can write

$$\underset{1\,4}{X} D_x\left(\frac{\psi_x}{\psi}\right) = D_x \underset{1\,4}{X}\left(\frac{\psi_x}{\psi}\right) = D_x\left(\frac{\mathrm{i}}{2}\right) = 0.$$

Hence, the expression

$$\omega_4 = D_x\left(\frac{\psi_x}{\psi}\right) = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}$$

is the second-order differential invariant of algebra (4.2).

Because of the reality of the invariant differentiation operators we can write that

$$\omega_4^* = \frac{\psi_{xx}^*}{\psi^*} - \frac{\psi^{*2}_x}{\psi^{*2}}$$

is also the differential invariant of algebra (4.2).

Further we obtain

$$\omega_5 = Q_2\left(\frac{\psi_x}{\psi}\right) = \psi\psi^* \cdot \left(\frac{\psi_{tx}\psi - \psi_x\psi_t}{\psi^2}\right) + \mathrm{i}(\psi\psi_x^* - \psi_x\psi^*) \cdot \left(\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}\right),$$

$$\omega_5^* = \overline{\omega_5}, \qquad \omega_6 = Q_2(\omega_2), \qquad \omega_6^* = \overline{Q_2(\omega_2)}.$$

The invariants $\omega_k, \omega_k^*, k = 4, 5, 6$ are functionally independent (over $\mathbb{R}$), because they contain different derivatives of the second order of ψ and ψ^*. Hence, $\omega_i, \omega_i^*, i = 0, \ldots, 6$, form the basis of the second-order general invariant of algebra (4.2) and all second-order differential invariants can be obtained from first order ones.

Remark 4.2. Note that one cannot get $\omega_k, \omega_k^*, k = 4, 5, 6$, by invariant differentiation and functional operations using only ω_0, but it is possible to obtain them using $\omega_k, k = 0, 1, 2, 3$.

Moreover, separating the variables ψ and ψ^* after invariant differentiation the previously studied Schrödinger equation can be deduced, as well as its differential invariants of second order. This fact shows the generality of the Schrödinger equation and the magnitude of the associated conservation laws. Further, an infinite sequence of integrals of motion can be constructed. It is therefore possible to establish integrals containing derivatives of arbitrary order.

We show the construction of nonlinear Schrödinger equation using obtained invariants. To this end, consider a new invariant Ω:

$$\Omega = \mathrm{i} \cdot \omega_2 + \omega_4 = \mathrm{i}\frac{\psi_t}{\psi} + \frac{\psi_{xx}}{\psi}.$$

Now we take invariant equation

$$\Omega = F(\omega_0),$$

where F is an arbitrary function. Hence,

$$\mathrm{i}\frac{\psi_t}{\psi} + \frac{\psi_{xx}}{\psi} = F(|\psi|^2).$$

Now multiplying by ψ and putting $F(|\psi|^2) = -W(|\psi|)$ we obtain the studied nonlinear Schrödinger equation $\mathrm{i}\psi_t + \psi_{xx} + W(|\psi|) \cdot \psi = 0$.

4.2. The case $\mathrm{i}\psi_t + \psi_{xx} + |\psi|^2\psi = 0$

Now let consider the nonlinear Schrödinger equation of the form

$$\mathrm{i}\psi_t + \psi_{xx} + |\psi|^2\psi = 0. \tag{4.8}$$

The generalization of this equation with functional coefficients and its differential invariants were investigated by Senthilvelan, Torrisi and Valenti in [9] by using equivalence transformations. They showed that this generalized Schrödinger equation admits an infinite-dimensional equivalence symmetry algebra. The corresponding Lie algebra of point symmetries is also infinite. However, in order to deduce the main physical properties we only need to work with a finite quantity.

Fact 4.1. *The complete symmetry algebra of equation (4.8) is infinite dimensional and has the infinitesimal generators of the form*

$$X_{\alpha_1} = \alpha_1(t)\psi\partial_\psi - \left[\frac{\mathrm{i}}{\psi} \cdot \alpha_1'(t) + \psi^*\alpha_1(t)\right]\partial_{\psi^*}, \tag{4.9a}$$

$$X_{\alpha_2} = \alpha_2(t)\partial_t + \frac{1}{2}x\alpha_2'(t)\partial_x + \frac{\mathrm{i}}{8}x^2\psi\alpha_2''(t)\partial_\psi$$
$$+ \frac{1}{8\psi^2}[x^2\psi\alpha_2'''(t) - 2\mathrm{i}\psi\alpha_2''(t) - 8\psi^2\psi^*\alpha_2'(t) - \mathrm{i}x^2\psi^2\psi^*\alpha_2''(t)]\partial_{\psi^*}, \tag{4.9b}$$

$$X_{\alpha_3} = \alpha_3(t,x)\partial_\psi - \frac{1}{\psi^2}[\mathrm{i}\alpha_{3t}'(t,x) + \alpha_{3xx}''(t,x) + 2\psi\psi^*\alpha_3(t,x)]\partial_{\psi^*}, \tag{4.9c}$$

$$X_{\alpha_4} = \alpha_4(t)\partial_x + \frac{\mathrm{i}}{2}x\psi\alpha_4'(t)\partial_\psi + \frac{1}{2}x\left[\frac{\alpha_4''(t)}{\psi} - \mathrm{i}\psi^*\alpha_4'(t)\right]\partial_{\psi^*}, \tag{4.9d}$$

where $\alpha_1(t), \alpha_2(t), \alpha_3(t,x), \alpha_4(t)$ *are the arbitrary real functions.*

Here there is an asymmetry with respect to ψ and ψ^*. The coefficient at ∂_{ψ^*} has such a form, in order to hold the invariancy condition. However, if we consider the system of equations (the equation in question and the conjugated equation)

$$\begin{cases} \mathrm{i}\psi_t + \psi_{xx} + |\psi|^2\psi = 0 \\ -\mathrm{i}\psi_t^* + \psi_{xx}^* + |\psi|^2\psi^* = 0, \end{cases} \tag{4.10}$$

then we obtain only the five-dimensional, solvable Lie algebra of symmetry of this system:

$$\begin{aligned} &X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = \psi\partial_\psi - \psi^*\partial_{\psi^*}, \\ &X_4 = t\partial_x + \frac{\mathrm{i}}{2}x(\psi\partial_\psi - \psi^*\partial_{\psi^*}), \qquad X_5 = 2t\partial_t + x\partial_x - \psi\partial_\psi - \psi^*\partial_{\psi^*} \end{aligned} \tag{4.11}$$

with commutation relations as in (4.2) and with respect to the fifth generator we have:

$$[X_1, X_5] = 2X_1, \qquad [X_2, X_5] = X_2, \qquad [X_3, X_5] = 0, \qquad [X_4, X_5] = -X_4.$$

Such consideration is very natural, because if the Schrödinger equation is fulfilled, then the conjugated equation ought also to be fulfilled. In algebra (4.11) we have four operators from algebra (4.2) and new operator X_5. Note that these operators satisfy the condition $X = \overline{X}$ and one can obtain them from the whole algebra of symmetry of equation (4.8) requiring this condition.

We find the differential invariants and the invariant differentiation operators for algebra (4.11).

Note that this algebra has not differential invariants of the order zero. Indeed

$$R(0) = 2 + 2 \cdot \binom{2+0}{2} - 4 = 0.$$

There are three first-order basic differential invariants, because $R(1) = 2 + 2 \cdot \binom{2+1}{2} - 5 = 3$. One can obtain them by functional operations on invariants of algebra (4.2):

$$\begin{aligned} \widetilde{\omega}_1 &= \frac{\omega_1}{\sqrt{\omega_0}} = \frac{\psi_x}{|\psi|\psi} + \frac{\psi_x^*}{|\psi|\psi^*}, \\ \widetilde{\omega}_2 &= \frac{\omega_2}{\omega_0} = \frac{\psi_t}{|\psi|^2\psi} - \mathrm{i} \cdot \left(\frac{\psi_x}{|\psi|\psi}\right)^2, \\ \widetilde{\omega}_3 &= \frac{\omega_3}{\omega_0} = \frac{\psi_t^*}{|\psi|^2\psi^*} + \mathrm{i} \cdot \left(\frac{\psi_x^*}{|\psi|\psi^*}\right)^2. \end{aligned}$$

By analogy we show that it is possible to construct the six second-order basic differential invariants using invariants of algebra (4.2):

$$\begin{aligned} &\widetilde{\omega}_4 = \frac{\omega_4}{|\psi|^2} = \frac{1}{|\psi|^2\psi^2}(\psi_{xx}\psi - \psi_x^2), && \widetilde{\omega}_4^* = \overline{\widetilde{\omega}_4}, \\ &\widetilde{\omega}_5 = \frac{\omega_5}{|\psi|^5} = \frac{\psi_{tx}\psi - \psi_x\psi_t}{|\psi|^3\psi^2} + \frac{\mathrm{i}(\psi\psi_x^* - \psi_x\psi^*)}{|\psi^5|} \cdot \left(\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}\right), && \widetilde{\omega}_5^* = \overline{\widetilde{\omega}_5}, \\ &\widetilde{\omega}_6 = \frac{\omega_6}{|\psi|^6}, && \widetilde{\omega}_6^* = \overline{\widetilde{\omega}_6}. \end{aligned}$$

It is easily seen that $\widetilde{\omega}_k, \widetilde{\omega}_k^*$ are functionally independent over $\mathbb{R}$, because they contain the second-order derivatives independently. It is also easy to check that one can obtain the new second-order basic invariants from the first order ones by new invariant differentiation and functional operations.

The new invariant differentiation operators for algebra (4.11) are the following:

$$\widetilde{Q}_1 = \frac{1}{|\psi|} \cdot Q_1 = \frac{1}{|\psi|}D_x, \qquad \widetilde{Q}_2 = \frac{1}{|\psi|^4} \cdot Q_2 = \frac{1}{|\psi|^2}D_t + \frac{\mathrm{i}(\psi\psi_x^* - \psi_x\psi^*)}{|\psi|^4}D_x.$$

We show the invariant construction of the studied Schrödinger equation by using the first-order differential invariants. First, we construct $\widetilde{\omega}_4$:

$$\Omega_1 = \widetilde{Q}_1(\widetilde{\omega}_1) = \frac{\psi_{xx}\psi - \psi_x^2}{|\psi|^2\psi^2} - \frac{1}{2|\psi|^3}(\psi_x\psi^* + \psi\psi_x^*)\cdot\widetilde{\omega}_2.$$

Because of the invariancy of the last expression we get $\widetilde{\omega}_4$ by functional operations.

Now we take the invariant equality:

$$\mathrm{i}\cdot\widetilde{\omega}_2 + \widetilde{\omega}_4 + 1 = 0 \tag{4.12}$$

and we have

$$\mathrm{i}\frac{\psi_t}{|\psi|^2\psi} + \left(\frac{\psi_x}{|\psi|\psi}\right)^2 + \frac{1}{|\psi|^2\psi^2}\left(\psi_{xx}\psi - \psi_x^2\right) + 1 = \mathrm{i}\frac{\psi_t}{|\psi|^2\psi} + \frac{\psi_{xx}\psi}{|\psi|^2\psi^2} + 1 = 0,$$

so the studied Schrödinger equation. By conjugation of (4.12) we obtain an analogous identity.

Remark 4.3. Note that in general for the operators X_{α_k} from algebra (4.9) the expression $\psi\psi^*$ is not an invariant. Hence, these operators generate a non-physical symmetry transformations of the Schrödinger equation. It appears that the condition $X = \overline{X}$ is also not sufficient to the conservation of the function $\psi\psi^*$. Indeed, in algebra (4.11) there is the operator X_5, for which the function $\psi\psi^*$ is not an invariant.

Finally, we obtain that only four operators from algebra (4.2) generate the physical symmetry transformations, conservating the function $\psi\psi^*$.

5. Conclusions

The Tresse theorem allows one to describe the structure of all invariants and differential invariants of a given Lie group by constructing a basis of invariants. Moreover, one can state whether or not some invariant is fundamental, meaning whether it can be obtained from another invariants by invariant differentiation and functional operations.

Such fundamental values, forming a basis of invariants, describe the basic conservation laws in geometry and physics. It is important that for any Lie group with a finite number of group parameters there exists a finite basis of invariants. Physically, it means that for some space with a given symmetry there exists a finite set of basic conservation laws. Another laws can be deduced from this set. For example, for the rotation and Lorentz groups in examples (3.1), (3.2), (3.3) we show that the basis consists only of the zeroth-order invariants, and for the symmetry group of the nonlinear Schrödinger equation the basis also contains invariants of the first order. Additionally, the operators of the invariant differentiation give invariant vector fields, important in applications and in theoretical physics.

This approach enables one to find perhaps more fundamental or general rules and laws then a studied equation. It has been pointed out that the invariance of $\psi\psi^*$ determines the considered Schrödinger equations. Moreover, it enables one to construct some new invariant equations, for example (4.4), (4.5), which may also describe fundamental physical rules. Additionally, it is useful by the classification of PDEs with symmetry point of view, giving the invariant form of studied equations.

It is important that the studied general Schrödinger equation is a consequence of fundamental invariant value $\psi\psi^*$, obtained by invariant differentiation and separation of the variables ψ, ψ^*. This fact shows the generality of the Schrödinger equation and the magnitude of the associated conservation laws.

It is likely that the approach presented in this work is also valid for equivalence transformations (see, e.g., [9, 12, 13]). For equations of these classes, differential invariants and equivalence properties can be analysed in a unified manner, and new interpretations of the involved quantities and observables could arise. Work in this direction will be the subject of further investigation.

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